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Randomly amplified discrete Langevin systems

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(Received 10 November 1998)

A discrete stochastic process involving random amplification with additive noise is studied analytically. If the non-negative random amplification factor b is such that $\langle b^{\beta}\rangle=1$, where β is any positive noninteger, then the steady state probability density function for the process will have power law tails of the form $p(x)\sim 1/x^{\beta+1}$. This is a generalization of recent results for $0<\beta<2$ obtained by Takayasu, Sato, and Takayasu [Phys. Rev. Lett. **79**, 966 (1997)]. It is shown that the power spectrum of the time series x becomes Lorentzian, even when $1<\beta<2$, i.e., in the case of divergent variance. [S1063-651X(99)10306-4]

PACS number(s): 05.40.−a, 02.50.−r, 05.70.Ln, 61.10.Eq

A power law behavior of the distribution function is widely observed in nature [1]. Recently, Takayasu, Sato, and Takayasu presented a general mechanism leading to the power law distribution [2]. They analyzed a discrete stochastic process which involves random amplification together with additive external noise. They clarified necessary and sufficient conditions to realize a steady power law fluctuation with divergent variance using a discrete version of the linear Langevin equation expressed as

$$x(t+1)=b(t)x(t)+f(t), \tag{1}$$

where $f(t)$ represents a random additive noise, and $b(t)$ is a non-negative stochastic coefficient. They derived the following time evolution equation for the characteristic function $Z(\rho,t)$, which is the Fourier transform of the probability density $p(x,t)$:

$$Z(\rho,t+1)=\int_0^{\infty}W(b)Z(b\rho,t)db\,\Phi(\rho), \tag{2}$$

where $W(b)$ is the probability density of $b(t)$, and $\Phi(\rho)$ is the characteristic function for $f(t)$. They showed that when $\langle b^{\beta}\rangle=1$ holds for $0<\beta<2$, the second moment $\langle x^2(t)\rangle$ diverges as $t\to\infty$, but Eq. (2) has a unique steady and stable solution

$$\lim_{t\to\infty} Z(\rho,t)\equiv Z(\rho)=1-\text{const}\times|\rho|^{\beta}+\cdots, \tag{3}$$

which yields the power law tails in the steady probability density

$$\lim_{t\to\infty} p(x,t)\equiv p(x)\sim 1/x^{\beta+1}, \tag{4}$$

or equivalently, the cumulative distribution

$$P(\geqslant|x|)\sim 1/x^{\beta}. \tag{5}$$

They also made numerical simulations of Eq. (1) by employing a discrete exponential distribution for $W(b)$, and showed that the theoretical estimate of the relation between β and the parameters specifying $W(b)$ [Eq. (15) in Ref. [2]] nicely fits with the simulation ''even out of the range of applicability, $\beta>2$.'' They stated that ''the reason for this lucky coincidence is not clear,'' although they pointed out at the same time that the power law distribution tails are a generic property of Eq. (1) [2]. In this Brief Report, the following two statements will be presented.

(a) The theory of Ref. [2] can be straightforwardly extended for $\beta>2$: If $\langle b^{\beta}\rangle=1$ holds for a positive noninteger β, then there exists a unique steady and stable solution of Eq. (2),

$$Z(\rho)=\sum_{m=0}^{n}A_{2m}(-1)^{2m}\rho^{2m}-C|\rho|^{\beta}+O(\rho^{2n+2}), \tag{6}$$

where $2n$ is the largest even number that is smaller than β. This $Z(\rho)$ leads to $p(x)\sim 1/x^{\beta+1}$.

(b) When $\langle b^{\beta}\rangle=1$ for a noninteger β between 1 and 2, the power spectral density (PSD) of $x(t)$ is Lorentzian, increasing with the observation time T as

$$S(\omega,T)\sim\frac{2}{T}\frac{x_0^2}{\ln\langle b^2\rangle}\frac{(1/\tau_1)\langle b^2\rangle^T}{(1/\tau_1)^2+\omega^2}\quad\text{for } T\gg 1, \tag{7}$$

where

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$$x_0^2 \equiv \langle x^2(0)\rangle + \frac{\langle f^2\rangle}{\langle b^2\rangle - 1}, \tag{8}$$

$$\tau_1 = \frac{1}{\ln\langle b^2\rangle + \ln[1/\langle b\rangle]}. \tag{9}$$

From statement (a), ''the coincidence'' found in Ref. [2] is naturally understandable. To prove (a), we assume the following form for $Z(\rho)$:

$$Z(\rho) = \sum_{n=0}^{\infty} a_n \rho^n + |\rho|^\beta \sum_{n=0}^{\infty} c_n \rho^n, \quad a_0 \equiv 1, \tag{10}$$

and substitute it into Eq. (2) in the limit $t \to \infty$. If $\Phi(\rho)$ is an even function [i.e., the distribution function of $f(t)$ is symmetric as assumed in Ref. [2]], we can first prove that $a_1 = 0$ because $\langle b\rangle \neq 1$. Also, $c_1 = 0$ because $\langle b^{\beta+1}\rangle \neq 1$. Thanks to $a_{2m-1} = 0$ and $\langle b^{2m+1}\rangle \neq 1$, $a_{2m+1} = 0$ is derived. Similarly, $c_{2m-1} = 0$ and $\langle b^{\beta+2m+1}\rangle \neq 1$ yield $c_{2m+1} = 0$. We can thus prove that a_n and c_n in Eq. (10) vanish for all odd numbers n, i.e., Eq. (6) holds. [Note that the nth moment $\langle x^n(t)\rangle$ with $n > \beta$ diverges not only for even number n but also for odd number which corresponds to the vanishing coefficient a_n.] Taking exactly the same procedures as in Ref. [2], we can prove that this solution is unique and stable. In case of $\beta > 2$, we have a finite variance but higher order moments: $\langle x^n(t)\rangle$, with $n > \beta$, diverge as $t \to \infty$.

To derive the probability density $p(x)$, we only need to assume that all kth derivatives of $Z(\rho)$ satisfy the boundary condition

$$\lim_{\rho \to \pm\infty} d^k Z(\rho)/d\rho^k = 0. \tag{11}$$

Using Eq. (11), we can partially integrate the expression

$$p(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\rho} Z(\rho) d\rho \tag{12}$$

$[\beta] + 1$ times, where $[\beta]$ is the largest integer that is smaller than β. Thus we obtain the asymptotic expansion as

$$p(x) \sim |x|^{-(\beta+1)} \int_{-\infty}^{\infty} e^{-i\xi} |\xi|^{\beta - [\beta] - 1} d\xi$$
$$\sim |x|^{-(\beta+1)} \Gamma(\beta - [\beta]), \tag{13}$$

where Γ is the γ function.

To prove statement (b), we note that the two-time correlation function is rigorously obtained from Eq. (1):

$$\phi(\tau, t) \equiv \langle x(t+\tau) x(t)\rangle = \langle x^2(t)\rangle \langle b\rangle^\tau, \tag{14}$$

where

$$\langle x^2(t)\rangle = \langle b^2\rangle^t \langle x^2(0)\rangle + \frac{1 - \langle b^2\rangle^t}{1 - \langle b^2\rangle} \langle f^2\rangle. \tag{15}$$

If $1 < \beta < 2$, we have a relation $0 < \langle b\rangle < 1 < \langle b^2\rangle$ because the function $G(\gamma) \equiv \langle b^\gamma\rangle$ satisfies $G(0) = 1$ and $G''(\gamma) > 0$ [2]. Then ϕ increases with t, but decays with τ as $\sim e^{-\tau/\tau_0}$ for any fixed value of t (Debye-type relaxation), with the relaxation time

$$\tau_0 = \frac{1}{\ln[1/\langle b\rangle]}. \tag{16}$$

Since the correlation function depends on both τ and t, the Wiener-Khinchin relation cannot be used to obtain the PSD. Defining the PSD which depends on the observation time T as

$$S(\omega, T) \equiv \left\langle \left| \int_0^T e^{i\omega t} x(t) dt \right|^2 \right\rangle \Big/ T = 2\,\mathrm{Re}\left\{ \int_0^T d\tau \int_0^{T-\tau} dt\, e^{i\omega\tau} \langle x(t+\tau) x(t)\rangle \right\} \Big/ T, \tag{17}$$

and using $\phi(\tau, t)$ obtained above, we arrive at expression (7). The spectrum is of $1/f^2$ type for $f \gg 1/\tau_1$ and flat for $f \ll 1/\tau_1$. Equation (7) implies that the power increases exponentially with the observation time T, which corresponds to the divergent behavior of the variance $\langle x^2(t)\rangle$. [We have neglected the case $0 < \beta < 1$, where even the average of x diverges as $\langle x(t)\rangle = \langle b\rangle^t \langle x(0)\rangle$ because $\langle b\rangle > 1$.]

When $\beta > 2$, both $0 < \langle b\rangle < 1$ and $0 < \langle b^2\rangle < 1$ hold, and results are rather trivial:

$$\langle x^2\rangle \equiv \lim_{t \to \infty} \langle x^2(t)\rangle = \frac{1}{1 - \langle b^2\rangle} \langle f^2\rangle, \tag{18}$$

$$\phi(\tau) \equiv \lim_{t \to \infty} \phi(\tau, t) = \langle x^2\rangle \langle b\rangle^\tau, \tag{19}$$

$$S(\omega) \equiv \lim_{T \to \infty} S(\omega, T) = 2\langle x^2\rangle \frac{(1/\tau_0)}{(1/\tau_0)^2 + \omega^2}. \tag{20}$$

Thus, as far as the PSD is measured, we cannot observe any singular aspect, higher order singularities being hidden.

The stochastic process described by Eq. (1) generally leads to the power law behavior $p(x) \sim 1/x^{\beta+1}$, while it also yields a Lorentzian spectrum $S(\omega) \propto 1/[(1/\tau)^2 + \omega^2]$. A colored noise, or $1/f^\alpha$ fluctuation, whose PSD is proportional to $1/\omega^\alpha$, has attracted much attention since $1/f$ noise was discovered several decades ago [3]. Such a power law behavior of the PSD is also observed widely in nature, and these two power laws, one in the probability density and the other in the PSD, are sometimes discussed together [2]. Therefore it is interesting to know whether an extremely long time scale τ can be involved in the present stochastic process. Because, in that case, the observation time T, which relates to the low frequency cutoff $\omega_0 = 2\pi/T$, cannot reach this time scale, then a $1/f^2$ fluctuation; that is, $S(\omega) \sim 1/\omega^2$ (for $\omega \gg \omega_0$) is observed *practically*.

One can immediately see that the time constant τ_0 or τ_1 becomes large in very limited cases. First, the average of b should be close to unity, i.e., $\langle b \rangle = 1 - \epsilon$ with $0 < \epsilon \ll 1$. Then τ_0 becomes $\sim 1/\epsilon \gg 1$. Furthermore, in the case of $\beta > 2$, we need $\langle b^2 \rangle$ smaller than unity, while in the case of $1 < \beta < 2$, the condition $\langle b^2 \rangle = 1 + \delta$ with $0 < \delta \ll 1$ is necessary. In the lat-ter case, we obtain $\tau_1 \sim 1/(\epsilon + \delta) \gg 1$. The exponential or Poisson distribution for $W(b)$ does not lead to such a long time constant. One example of large τ_1 is obtained by choosing $W(b)$ to be a narrowly peaked distribution having an average which is slightly smaller than unity and a second moment slightly larger than unity.

As pointed out above, it should be noted that a stochastic process whose stationary density function has power law tails will not necessarily exhibit a power law behavior in the PSD.

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